Summary

Traditional yield analysis is not enough to capture the complete risk and return characteristics of bonds with embedded options. Traditional yield analysis estimates the incremental return over a benchmark security such as a Treasury bond, but bonds with embedded options have uncertain cash flows due to the uncertainty in future interest rates. It is essential for this additional risk to be priced as accurate as possible in order to isolate the option risk from the traditional fixed income risks such as interest rate, liquidity and credit risk. For example, if all the incremental return versus a benchmark security is due to the riskiness of the embedded option, then it may not make sense to buy that bond versus the benchmark. In this case the investor is not being compensated for the traditional fixed income risks.

Option pricing theory provides the tools to price bonds with embedded options in a no-arbitrage framework that reflects the current prices of bonds across all maturities. In turn, we can isolate the option risk and estimate the incremental return over the benchmark yield curve apart from the embedded optionality. Finally, we can calculate more accurate measurements of interest rate risk in a particular bond.

All financial models are simplifications of reality, and there is no exception with regards to the valuation model presented in this paper. The model’s tractability lies in its simplified structure and calibration to current prices in the bond market.

Introduction

Fixed income securities are generally represented by a corresponding yield measure, specifically the annualized yield to maturity. The yield to maturity, denoted as \( r \) in equation 1.1, is the discount factor equating the sum of the present value of all future cash flows \( (CF_i) \) at time \( i \) of the bond to its current market price.\(^1\) An investor can then compare the yield spread, or the difference between this yield and the yield on a risk-free US Treasury bond, to gain a basic understanding of the return characteristics of this particular bond.

\[
\text{Bond Price} = \sum_{i=1}^{n} \frac{CF_i}{(1+r)^i}
\]  

(1.1)

The yield to maturity is often used as the estimate for the annual return of a fixed maturity bond (bullet), but there are some shortcomings with this internal rate of return. According to 1.1, all income generated (i.e. coupon payments) is assumed to be reinvested at this internal rate of return, \( r \). This will only hold true if the yield curve is flat and does not change throughout the life of the bond. Interest rates can fluctuate up or down throughout the lifetime of a fixed income security, and although some models attempt to predict future interest rates, the evidence is that future interest rates are unpredictable. Second, many bonds carry prepayment

\(^1\) For simplicity, compounding has been left out of the equation.
options allowing the borrower to retire the loan prior to maturity. The yield to maturity assumes that the bond will last to its stated maturity. These limitations call for more sophisticated techniques to valuing these complicated structures.

In Financial Economics it is customary to decompose a complex security into a portfolio of two or more underlying simpler securities. Under the law of one price, if we can value the portfolio of simpler securities, then we can value the more complex security. For example, the future expected payoffs of a European call option on a stock can be replicated by a portfolio consisting of a long position in the underlying stock and a short position in a risk-free bond with maturity equal to the life of the option contract. The prices should be equal; otherwise there exists an arbitrage opportunity in the market for these securities. We can apply this framework to determine an arbitrage-free price for the call option using only the prices of the underlying stock and bond positions (Black and Scholes 1973).

A callable bond can be viewed as holding a portfolio of two securities: a long position in a bullet bond with the same nominal maturity as the callable issue and a short position in a call option on the bond, as shown in equation 1.2. The option is held as a short position because the lender sells the call option to the borrower. The borrower then owns the right to redeem the issue prior to the stated maturity at a specific strike price and time in the future.

\[ \text{callable bond} = \text{bullet bond} - \text{call option} \quad (1.2) \]

The price of the bullet bond is readily available in the market place, but the option price is much less transparent since there isn’t a centralized exchange for these types of options as with equity securities. The Black-Scholes option pricing model was the valuation model of choice twenty to thirty years ago, but the original model did not translate well to the intricacies of bond options. Modeling the underlying variable of a bond option, i.e. the interest rate, is a bit more complex when considering the entire term structure of interest rates. Consequently if the option price is inaccurate, then the callable bond price will not reflect the true compensation (in higher yield) to the lender for taking on the additional option risk as shown in equation 1.3.

\[ \text{yield}_{\text{callable}} = \text{yield}_{\text{bullet}} + \text{spread}_{\text{option}} \quad (1.3) \]

Another limitation, after examining 1.3, is one cannot isolate any yield spread in the callable bond due to duration, liquidity or credit risk. If the yield spread is purely due to the riskiness of selling the embedded call option, then it may not make sense to hold the callable bond. In this case the investor is not being compensated for traditional fixed income risks such as credit, liquidity or duration risk as shown in equation 1.4.
\[ \text{yield}_{\text{callable}} = \text{yield}_{\text{bullet}} + \text{spread}_{\text{option}} + \text{spread}_{\text{other risks}} \]  \hfill (1.4)

The following paper describes a well-known framework for valuing bonds with embedded options, specifically callable bonds. Section 2 defines the inputs needed for the valuation model. Section 3 explains the underlying theoretical assumptions of the model. Section 4 outlines the steps for valuing a callable bond and its option adjusted spread (OAS) and section 5 concludes with derivations of the interest rate risk measures of duration and convexity for these securities.

Section 2

There will be two main parameters to the interest rate model: an interest rate curve and interest rate volatility curve. The interest rate curve input is the current benchmark forward rate curve. Forward rates are the building blocks in all fixed income securities. A forward rate is an observable interest rate that an investor can essentially lock in today for a specific time period in the future.

The interest rate model introduced here describes the evolution of the short interest rate forward in time. In order to prevent arbitrage the forward rates are calibrated so that discounting a future cash flow along a path of forward short rates is equivalent to discounting the same cash flow at the current spot rate as in equation 2.1 in its continuous time version, where \( P(t,T) \) is the price at time \( t \) of a pure discount bond with spot rate \( r \) returning $1 at time \( T \).

\[ P(t,T) = 1 \exp \left\{ -r(t)(T-t) \right\} \]  \hfill (2.1)

This framework is popular because the model utilizes observed interest rates in the market to value securities rather than generate a yield curve based on a general equilibrium model that may or may not depict current bond prices in the market.

We first begin with a benchmark interest rate curve such as the US Treasury curve. This yield curve is a collection of bonds across a variety of maturities priced at par ($100) with semi-annual coupon payments. One shortcoming using this yield curve is that similar periodic cash flows are found to be discounted at different interest rates. For example, suppose the benchmark curve currently reports the yield on a six-month and a one-year security at 1% and 2% respectively. Both coupon-bearing securities have a cash flow payment in six months, yet it is unclear what the appropriate discount rate is to present value these six-month cash flows. This problem is

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2 For example, if one defines the short rate as the 6-month Treasury bill, then the forward curve for the 6-month Treasury bill could be the current 6-month T-bill, the 6-month T-bill 6-months forward, the 6-month T-bill 12-months forward, the 6-month T-bill 18-months forward and so on...
corrected by treating each cash flow as a zero-coupon bond and calculating zero-coupon spot rates from these adjusted cash flows. First, there is only one cash flow with the six-month security, and therefore the current six-month benchmark yield is equivalent to the six-month zero-coupon spot rate ($r_{6m}$). This is calculated explicitly in equation 2.2.

$$PV_{6m} = \frac{CF_{6m}}{(1 + r_{6m}/m)^{m}} \quad (2.2)$$

$$\left(1 + \frac{r_{6m}}{2}\right)^{0.5 \times 2} = \frac{100.50}{100}$$

$$\left(1 + \frac{r_{6m}}{2}\right) = 1.0050$$

$$r_{6m} = 0.01$$

$PV$ = Present Value of cash flow; $CF_i$ = cash flow at time $i$; $t$ = time to cash flow; $m$ = number of coupon payments per year

Once the initial spot rate is calculated we can treat each future cash flow (coupon and/or principal payments) of the security as a zero-coupon bond, and discount the cash flow by the appropriate zero-coupon spot rate. For example, the one-year bond priced at $100 with a coupon of 2% can be decomposed into its simpler parts: the sum of the present value of two zero-coupon securities maturing in six months and one year with cash flows of $1 and $101 respectively. The aim is to derive an adjusted present value of the one-year bond in order to calculate the one-year zero coupon spot rate. This is done by discounting the six-month cash flow ($1) at the six-month spot rate and subtracting it from the par price of the one-year bond as in equation 2.3.
\[ PV_{6m} = \frac{CF_{6m}}{(1 + sr_{6m}/m)^{tm}} \]

\[ = \frac{1}{(1 + 0.01/2)^{0.5*2}} \]

\[ = 0.99502 \]

Adjusted Bond Price_{1y} = Bond Price_{1y} - PV_{6m} \hspace{1cm} (2.3)

\[ = 100 - 0.99502 \]

\[ = 99.004975 \]

Since both the future value and the present value of the one-year zero-coupon bond are known, we can use some Algebra again to solve for the one-year spot rate as shown in equation 2.4. This iterative process continues for the other maturities until a unique set of zero-coupon spot rates is established across the entire yield curve (essentially equation 2.1 for each maturity on the spot curve).

\[ FV = PV * \left(1 + sr_{1y}/m\right)^{tm} \]

\[ sr_{1y} = \left(\frac{tm}{\sqrt{FV/PV} - 1}\right) * m \hspace{1cm} (2.4) \]

\[ = \left(\frac{101}{\sqrt{99.004975} - 1}\right) * 2 \]

\[ sr_{1y} = 0.02005 \]

The final step is to construct the forward rate curve from the unique set of zero-coupon spot rates. Looking at a one-year time horizon, a lender has the option to invest in a one-year bond at the current one-year spot rate or invest in a six-month bond at the current six-month spot rate and then roll it into another six-month bond at maturity. It isn’t known what this future six-month spot rate will be, but other things equal, the returns on these two investments must be equivalent in order to prevent an arbitrage situation. By definition the initial implied forward rate (fr_1) is set equal to the initial spot rate (sr_1). Using another iterative process shown in
equation 2.5, we can solve for subsequent period forward rates until establishing the forward curve.

\[
(1 + \frac{fr_{i+1}}{2})^{2t_i} \times (1 + \frac{fr_{i+1}}{2})^{2t_i} = (1 + \frac{sr_{i+1}}{2})^{2t_i,1}
\]

(2.5)

\[
(1 + \frac{fr_{i+1}}{2}) = \frac{(1 + \frac{sr_{i+1}}{2})^{2t_i,1}}{(1 + fr_i)^{2t_i}}
\]

With the initial forward rate set at 1.00% the six-month forward rate in six months is calculated below as an example:

\[
(1 + \frac{fr_1}{2})^{2t_1} \times (1 + \frac{fr_1}{2})^{2t_1} = (1 + \frac{sr_{i+1}}{2})^{2t_i,1}
\]

\[
(1 + \frac{fr_z}{2}) = \frac{(1 + \frac{0.02005}{2})^2}{(1 + \frac{0.01}{2})}
\]

\[
(1 + \frac{fr_z}{2}) = 1.015075
\]

\[
fr_z = 0.03015
\]

Table 2.1 reports spot rate and forward rate curves going out to three years using a hypothetical benchmark on the run curve and these iterative processes.

<table>
<thead>
<tr>
<th>Maturity (years)</th>
<th>Benchmark Yield Curve</th>
<th>Spot Rate Curve</th>
<th>Forward Rate Curve</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1.000%</td>
<td>1.000%</td>
<td>1.000%</td>
</tr>
<tr>
<td>1.0</td>
<td>2.000%</td>
<td>2.005%</td>
<td>3.015%</td>
</tr>
<tr>
<td>1.5</td>
<td>3.000%</td>
<td>3.020%</td>
<td>5.066%</td>
</tr>
<tr>
<td>2.0</td>
<td>4.000%</td>
<td>4.051%</td>
<td>7.176%</td>
</tr>
<tr>
<td>2.5</td>
<td>5.000%</td>
<td>5.104%</td>
<td>9.372%</td>
</tr>
<tr>
<td>3.0</td>
<td>6.000%</td>
<td>6.187%</td>
<td>11.686%</td>
</tr>
</tbody>
</table>

Table 2.1. Interest Rate Curves
Section 3

We assume that the short interest rate \( r \), defined as the annualized one-period interest rate, follows a lognormal stochastic process as shown in its continuous-time form in equation 3.1 (a and \( b \) have the same meaning). According to the model, all bond prices and bond yields depend only on the short interest rate and therefore the expected returns on these bonds are equal over a specific time period. The power of this one-factor interest rate model, along with similar models of this nature, is that the output is a probability distribution of possible future short rates forward in time that is calibrated to the current yield curve as calculated in the previous section. These short rate paths can then be used to value bonds with embedded options and, in turn, option adjusted spreads and other risk measures.

\[
\begin{align*}
\dot{r} &= \mu(r,t)\, \Delta t + \sigma r \, \sigma Z(t) \tag{3.1a} \\
\dot{\ln}(r) &= \mu(\ln(r), t)\, \Delta t + \sigma \, \sigma Z(t) \tag{3.1b}
\end{align*}
\]

Equation 3.1 is the well-known geometric Brownian motion diffusion model and the basis for valuing many types of financial derivatives (\( \dot{r} \) is the symbol for change in continuous time). The equation states that over small time periods, changes in the short interest rate are a function of the current yield curve and the volatility of the short interest rate. Thus \( r \) evolves through time in a random manner. The source of this randomness is the normally distributed Weiner process, \( \sigma Z(t) \), with mean = 0 and variance = \( \sigma \Delta t \) (i.e. the incremental time change). The mean function \( \mu(r,t) \) is an implied drift function as determined by the current yield curve and \( \sigma \) is the standard deviation of the short interest rate. We implicitly make three strong assumptions using this model: 1) interest rates are lognormally distributed and therefore cannot go negative; interest rate changes are normally distributed, 2) the past history of interest rates is irrelevant to tomorrow’s expected interest rate (known as the martingale property) and 3) volatility is proportional to the level of interest rates. The addition of a mean reversion component to the drift function and/or making volatility a function of time turns 3.1 into the well-known Black-Derman-Toy interest rate model (1991).³

Pricing securities in continuous time as in 3.1 is often a difficult concept to grasp within the realm of the financial markets. Equation 3.2 outlines a more intuitive approach to modeling the short interest rate by using a binomial recombining tree in the spirit of the Cox-Ross-Rubenstein method for pricing stock options (1979). The convergence of the binomial distribution to the normal distribution in the limit as time changes go to zero allow us to view equation 3.1 in this

³ See the Appendix for an overview of the Black-Derman-Toy short rate model version. In this special case, the volatility in equation 3.1 would also be a function of time, \( t \). \( \dot{r} = \mu(r,t)\, \Delta t + \sigma(t) r \, \sigma Z(t) \)
discrete-time form (Δ signifies changes over discrete time increments, for example one month or one day).

\[
\Delta \ln(r) = \mu(\ln(r), t) \Delta t + \sigma \sqrt{\Delta t} \ast \varepsilon[-1, +1]
\]  
(3.2)

\( \varepsilon = Z_t - Z_{t+1} \) is the discrete-time version of \( \partial Z(t) \) in equation 3.1, also known as the random walk. \( \varepsilon \) follows a binomial distribution taking on values of -1 or 1 with probability of \( \frac{1}{2} \) (as in flipping an unbiased coin). According to 3.2, the short interest rate can only move up or down at any point in the model with equal probability. For example at current time, \( t = 0 \), the current short rate, \( r_0 \), can move up to \( r_u \) or down to \( r_d \), as shown in figure 3.1, by applying equation 3.2.

Equations 3.3a and 3.3b illustrate this mathematically.

\[
\ln(r_u) = \ln(r_0) + \mu(\ln(r_0), t) \Delta t + \sigma \sqrt{\Delta t}
\]  
(3.3a)

\[
\ln(r_d) = \ln(r_0) + \mu(\ln(r_0), t) \Delta t - \sigma \sqrt{\Delta t}
\]  
(3.3b)

Combining equations 3a and 3b yields a relationship between \( r_u \) and \( r_d \) at any time step in the interest rate tree. Equation 3.4, along with the calibration steps outlined in the next section, provide the basis for the evolution of the short rate in the binomial interest rate tree.

\[
\ln(r_u) - \ln(r_d) = 2 \sigma \sqrt{\Delta t}
\]

\[
\exp(\ln(r_u) - \ln(r_d)) = \exp(2 \sigma \sqrt{\Delta t})
\]

\[
\frac{r_u}{r_d} = \exp(2 \sigma \sqrt{\Delta t})
\]

\[
r_u = r_0 \exp(2 \sigma \sqrt{\Delta t})
\]  
(3.4)
Section 4

The following valuation of a callable bond will be conducted in a discrete-time binomial tree framework with six-month time step increments for demonstration purposes. These time steps are typically shortened to monthly or even daily time increments without sacrificing significant computational time.

Prior to walking through a concrete example some additional notation is helpful at this point to flag all the moving parts in the valuation model. Matrix notation is the easiest way to read interest rate and bond price trees. The time steps (t) denote the columns of the matrix [j] and each time step has a specific number of rows [i] based on the up and down movements of the short rate and the underlying binomial distribution. Therefore a specific position [i, j] in the matrix denotes a projected short interest rate (r[i, j]) or discounted cash flow (price[i, j]) in the binomial tree. This notation is presented in figure 4.1. For example, r[1,2] is the projected six-month short interest rates in six months based on the “up” move from the initial six-month rate at r[1,1].

\[ \begin{array}{c}
\text{price[1,1]} \\
\text{r[1,1]} \\
\end{array} \]

\[ \begin{array}{c}
\text{price[1,2]} \\
\text{r[1,2]} \\
\end{array} \]

\[ \begin{array}{c}
\text{price[2,2]} \\
\text{r[2,2]} \\
\end{array} \]

\[ \begin{array}{c}
\text{price[3,3]} \\
\text{r[3,3]} \\
\end{array} \]

\[ \begin{array}{c}
\text{price[1,3]} \\
\text{r[1,3]} \\
\end{array} \]

\[ \begin{array}{c}
\text{price[2,3]} \\
\text{r[2,3]} \\
\end{array} \]

\[ \begin{array}{c}
\text{price[3,3]} \\
\text{r[3,3]} \\
\end{array} \]

Figure 4.1. Matrix Notation of Binomial Tree

\[ t = 0 \quad t = 0.5 \quad t = 1.0 \quad t = 1.5 \]

Note: not all cells of the matrix will have actual values.
With the notation in place, figure 4.2 provides an example of a calibrated binomial interest rate tree with six-month time steps using the benchmark yield curve out to 1.5 years from table 2.1. The calibration involves three precise checks:

1. First, the short interest rate is projected forward to move up or down at any node based on equation 3.4 and assuming an annual constant volatility of $\sigma = 50\%$.
2. Second, in order to prevent arbitrage the present value of a zero-coupon bond discounted by the forward rates in the interest rate tree must equal the discounted present value calculated from the current spot rate for the same zero-coupon maturity.
3. Finally, the forward rates calculated in step 1 are never explicitly observed in the binomial tree, but the expected forward rate for a period in the binomial tree must equal the implied forward rate as calculated in section 2.

The calibration will be tested manually below, but an optimization technique such as the Newton-Raphson method is an efficient tool to test these conditions within a computer program at each time step in the interest rate model.

![Lognormal Binomial Interest Rate Tree]

Let’s examine the projected six-month forward rates in six months at time step $t = 0.5$. First, it is straightforward to check that equation 3.4 holds:
Second, we check the calibration to the spot rate, i.e. that a hypothetical one-year zero-coupon bond (at t=0) discounted by the forward rates at each node matches the present value discounted by the appropriate zero-coupon spot rate, in this case the one-year spot rate from table 2.1. The present value discounted by the spot rate is calculated as:

\[
\frac{1}{1 + \left(\frac{sr_{t+1}}{2}\right)^2} = \frac{1}{1 + \left(\frac{0.02005}{2}\right)^2}
\]

\[
= 0.98025
\]

The present value discounted from the forward rates from the interest rate tree is a multi-step process. The par value of the hypothetical zero-coupon bond is discounted by the forward rates at each node for t = 1 to determine the corresponding discounted forward prices at those nodes.

\[
\text{price}[1,2] = \frac{1}{(1 + \frac{r[1,2]}{2})^{2t}} \quad \text{price}[2,2] = \frac{1}{(1 + \frac{r[2,2]}{2})^{2t}}
\]

\[
\text{price}[1,2] = \frac{1}{(1 + \frac{0.04046}{2})} \quad \text{price}[2,2] = \frac{1}{(1 + \frac{0.01995}{2})}
\]

\[
= 0.98017 \quad \text{price}[2,2] = \frac{1}{(1 + \frac{0.01995}{2})}
\]

\[
= 0.99012
\]

Now, using risk-neutral pricing (i.e. probability of an up movement = probability of a down movement = 0.5), these prices at t = 1 are discounted by the initial forward rate. The present value discounted by the forward rates calculates to 0.98025, and therefore the no-arbitrage condition holds at t = 1.
\[ \frac{1}{2} \left( \text{price}[1,2] + \text{price}[2,2] \right) = \frac{1}{2} \left( 0.9801 + 0.9901 \right) \]
\[ = 0.98025 \]

Finally, we can check if the expected value of the forward rates at \( t = 1 \) match the forward rate 3.02% reported in table 2.1.

\[
\text{Expected Value} = \sum \text{Pr}[i,j] \cdot r[i,j], \text{ where}
\]
\[
\text{Pr}[i,j] = \text{the probability the short rate follows the path to node } [i,j]
\]
\[
0.0302 = \frac{1}{2} \times 0.04046 + \frac{1}{2} \times 0.01995
\]
\[
0.0302 = 0.0202 + 0.00998
\]
\[
0.0302 = 0.0302
\]

Since each of these conditions checked out we can move on to the next time step, \( t = 1.0 \), until the entire interest rate tree is constructed and calibrated to the current spot curve.

We now have the valuation tools to calculate the discounted cash flows of almost any fixed income structure and calibrate the model price to the market price. Using a backwards process beginning at the bond’s maturity, a corresponding price tree is constructed that mirrors the interest rate tree as shown in figure 4.2. The price at each node \([i,j]\) is the average of the discounted cash flows from the two connected nodes to the right, i.e. \( p[i, j+1] \) and \( p[i+1, j+1] \). The cash flows are discounted by the corresponding forward rates at each node in the tree according to equation 4.1.

\[
\text{price}[i,j] = \frac{1}{2} \left( \text{price}[i,j+1] + \text{price}[i+1,j+1] \right) \cdot \exp\{-r[i,j] \cdot dt\} \quad (4.1)
\]

This is best illustrated through our example: figures 4.3 and 4.4 display binomial price trees for hypothetical 1.5-year bonds with 3.25% coupon rates. The first bond is a fixed maturity bullet and the second bond is callable in one year at a strike price of 100, and therefore the price of the bond at those nodes in time step \( t = 1.0 \) is equal to the minimum of the discount price from equation 4.1 and the strike price. This is precisely what happens at node \([3,3]\) in figure 4.4 and
results in lower discounted cash flows in connected nodes (i.e. node [2, 2]). Notice that at \( t = 0 \) the price of the bullet bond in 4.3 is greater than the price of the callable bond in 4.4 bringing us full circle to the observation from equation 1.2: in equilibrium, the price of the callable bond should be less than the comparable bullet bond, compensating the lender with a higher yield for selling the option.

Figure 4.3. Binomial Price Tree of 1.5 Year Bullet Bond with 3.25% Coupon Rate
The prices generated in figures 4.3 and 4.4 are discounted off the calibrated spot curve. Typically these “model prices” will not match up to what a bond is priced at in the current market since the bonds trade at a yield spread over the benchmark curve. Therefore the model price is calibrated to the current market price of the bond by adjusting the short rates in the tree by a constant factor, or the option adjusted spread (OAS). Figure 4.5 calibrates the model price from figure 4.3 to a hypothetical market price of 100.00. The result is an OAS of 18.1 basis points over the calibrated forward rates in figure 4.2.
The value of the option is calculated by first valuing the hypothetical bullet bond from figure 4.3 at the 18.1bps OAS and subtracting the calibrated callable bond price at each node, i.e. rearranging equation 1.2: \( \text{call option} = \text{bullet} - \text{callable} \). If the difference is a negative value, then the option price is zero at that node, since the value of the option at each node is the maximum of 0 and the difference between the bond price and the strike price at that node: \( \max[0, \text{price}[i,j] - \text{strike}] \). This is shown in figures 4.6 and 4.7.
Figure 4.6. Binomial Price Tree for Hypothetical Noncallable Bond (OAS = 18.1bps)

Figure 4.7. Binomial Price Tree for embedded call option (OAS = 18.1bps)
Section 5

So far we have only discussed estimating the expected return portion of bonds with embedded options, but accurately measuring risk in these securities is just as important, if not more so. Specifically, we shift now to measuring interest rate risk or the duration and convexity of these securities.

Duration is a measure of the weighted average time for the recovery of the bond’s cash flows (including principal) in relation to the current price of the bond. The typical duration of a bond approximates the percentage change in the price of the bond for a 1% change in the internal rate of return or yield, assuming the cash flows of the bond do not change. This calculation works well for bullet bonds but not for bonds with embedded options since future cash flows can change depending on fluctuations in interest rates. For example, if interest rates fall 1%, then the probability of a callable bond being redeemed early in the near future necessarily increases. This possibility of early redemption shortens the average life of the bond and in turn its risk characteristics. A new duration measure called effective duration can be approximated for any security directly using the interest rate model introduced in the previous sections and Taylor expansion rules of approximating a differentiable function at a point.

Once the OAS is calculated for a bond with the embedded option each interest rate in the adjusted interest rate tree (as in figure 4.5) is shifted up and down by a specific amount (typically 0.10% or 10 basis points) to simulate a simple parallel shift of interest rates. The underlying cash flows are re-priced for each shift resulting in two new hypothetical bond prices. The percentage price change is then calculated for each up and down shift and finally a simple average of the two gives us the effective duration of the bond as shown below in equation 5.1.

\[
\text{Duration} = \sum_{i=1}^{n} t_i \left( \frac{C_F \exp(-rt)}{P} \right), \text{ where } P \text{ is the market price of the bond}
\]

\[
= t_1 \left( \frac{C_F \exp(-rt_1)}{P} \right) + t_2 \left( \frac{C_F \exp(-rt_2)}{P} \right) + \ldots + t_n \left( \frac{C_F \exp(-rt_n)}{P} \right)
\]

\[
= -\frac{\partial P}{\partial r} \frac{1}{P}
\]
Estimate percentage change in price for both up and down shifts in interest rates:

\[
P(r + \Delta r) = P(r) + \frac{\partial P}{\partial r} \Delta r
\]

\[
-P(r - \Delta r) = P(r) + \frac{\partial P}{\partial r} \Delta r
\]

\[
\frac{\partial P}{\partial r} \Delta r = P(r - \Delta r) - P(r + \Delta r)
\]

\[
\frac{\partial P}{\partial r} \Delta r = \frac{P(r) - P(r + \Delta r)}{\Delta r}
\]

Duration\textsubscript{1} = \frac{1}{P} \frac{\partial P}{\partial r} \Delta r = \frac{P(r) - P(r + \Delta r)}{P(r) \Delta r}

Duration\textsubscript{2} = \frac{1}{P} \frac{\partial P}{\partial r} \Delta r = \frac{P(r) - P(r - \Delta r)}{P(r) \Delta r}

Effective Duration is estimated by taking the average of the two duration measures above:

\[
\text{Effective Duration} = \frac{P(r - \Delta r) - P(r + \Delta r)}{2 \times P(r) \Delta r}
\]

The effective duration in 5.1 now approximates the percentage change in the bond price for a 1% parallel shift in interest rates when future cash flows are uncertain.

Once the calibration of the model price to the market price is completed as in section 4, the effective duration is straightforward. Keeping the OAS of 18.1 basis points constant, the interest rates in figure 4.1 are shifted up and down 10 basis points and new bond prices are calculated under the simulated parallel interest rate shift. Once these prices are calculated, the effective duration for the callable bond is calculated using equation 5.1 and turns out to be 1.33:

\[
\text{Effective Duration} = \frac{P(r - \Delta r) - P(r + \Delta r)}{2 \times P(r) \Delta r}
\]

\[
= \frac{100.133 - 99.867}{2 \times 100 \times 0.0010}
\]

\[
= 1.33
\]
The relationship between bond prices and interest rates is not a linear relationship, and therefore the effective duration measure in 5.1 is only a first approximation for bond price changes with respect to interest rate changes. The curvature of the bond price function is explained by the effective convexity measure, sometimes referred to as the second derivative of the bond price with respect to interest rates. Effective convexity can be estimated in a similar manner to the effective duration in 5.1 and its relationship to the second derivative of the bond price function with respect to the interest rate.\(^6\) Convexity explains the change in the bond price that cannot be explained by the effective duration.

**Effective Convexity calculation using the Second Order Taylor Approximation:**

\[
P(r + \Delta r) - P(r) = \frac{\partial^2 P}{\partial r^2} * \Delta r + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} * (\Delta r)^2
\]

(5.2)

\[
P(r - \Delta r) - P(r) = \frac{\partial^2 P}{\partial r^2} * (-\Delta r) + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} * (-\Delta r)^2
\]

(5.3)

**Adding 5.2 and 5.3 together and using the original definition for Convexity:**

\[
\frac{\partial^2 P}{\partial r^2} * (\Delta r)^2 = P(r - \Delta r) + P(r + \Delta r) - 2P(r)
\]

Effective Convexity = \(\frac{\partial^2 P}{\partial r^2} * \frac{1}{P}\)

(5.4)

**Effective Convexity = \(\frac{\partial^2 P}{\partial r^2} * \frac{1}{P} = \frac{P(r - \Delta r) + P(r + \Delta r) - 2P(r)}{P(r) * (\Delta r)^2}\)**

\(^6\) Convexity = \(\frac{\partial^2 P}{\partial r^2} * \frac{1}{P}\), where P is the market price of the bond.
Effective Convexity = \frac{P(r - \Delta r) + P(r + \Delta r) - 2P(r)}{P(r) * (\Delta r)^2}

= \frac{100.133 + 99.867 - 2*100}{100*0.0010^2}

= 2.34

The effective duration and convexity measures can be used to estimate the percentage price change of a callable bond for an instantaneous parallel shift in interest rates. We can demonstrate this using the same callable bond with a current market price of 100. Suppose we would like to estimate the percentage price change for a 100 basis point (1%) positive move in interest rates. Again, the new bond price can be approximated using the Taylor series expansion to the second degree as in equation 5.5 below.

\begin{equation}
P(r + \Delta r) = P(r) + \frac{\partial P}{\partial r} * \Delta r + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} * (\Delta r)^2, \text{ Taylor Series Expansion}
\end{equation}

since \( \frac{\partial P}{\partial r} = -\text{Duration} P(r) \) and \( \frac{\partial^2 P}{\partial r^2} = \text{Convexity} P(r) \)

\begin{equation}
P(r + \Delta r) = P(r) - \text{Duration} P(r) * \Delta r + \frac{1}{2} \text{Convexity} P(r) * (\Delta r)^2
\end{equation}

rearranging terms, we get:

\begin{equation}
\frac{P(r + \Delta r) - P(r)}{P(r)} = -\text{Duration} * \Delta r + \frac{1}{2} \text{Convexity} * (\Delta r)^2 \tag{5.5}
\end{equation}

Finally, plugging in the measures for duration and convexity from earlier, the estimated price change for a 100 basis point parallel shift is:

\begin{equation}
\frac{P(r + 0.01) - P(r)}{P(r)} = -1.33 * 0.01 + \frac{1}{2} * 2.34 * (0.01)^2
\end{equation}

= -1.318%
Conclusion

All financial models are simplifications of reality, and there is no exception with regards to the valuation model presented in this paper. The model’s tractability lies in its simplified structure and calibration to the current yield curve. The main output of the model, the option adjusted spread is a critical measure in determining the risk/reward tradeoff between many fixed income securities. The valuation model also calculates more accurate interest rate risk measures of effective duration and convexity by shifting the forward rates in the binomial tree and allowing for uncertainties in cash flows.
References


Appendix: Black-Derman-Toy

Fischer Black, Emanuel Derman and William Toy (BDT model) published a special version of the one-factor log-normal interest rate model back in 1991. The group had been working on and implementing the model at Goldman Sachs several years prior. BDT brought the original log-normal interest rate model a bit closer to the real world by introducing the volatility as a function of time in equation 3.1. It was well-known at the time that interest rate volatilities differed across the maturity spectrum and BDT built this into their interest model. The BDT one-factor short rate model includes simultaneous calibrations to the current yield curve and the current volatility curve bringing the model a little closer to the reality of the fixed income markets. Whereas the original log-normal model only allows for parallel moves in interest rates, BDT can model more complex changes in the shape of the yield curve. Most of today’s popular interest rate models such as Hull-White and Heath-Jarrow-Morton have their roots in the BDT model. For vanilla-type structures such as bonds with the embedded options, the BDT model remains a very popular tool for pricing these securities. The essence of the BDT model can be observed in equations A.1 and A.2 after making the volatility factor as a function of time in equation 3.1a and b from section 3. The power of this one-factor interest rate model, along with similar models of this nature, is that the output of the model is a probability distribution of possible future short rates forward in time that is calibrated to the current yield curve. These forward short rates can then be used to value bonds with embedded options and, in turn, option adjusted spreads.

\[
\hat{r} = \mu(r,t)\hat{t} + \sigma(r)\hat{Z}(t) \tag{A.1}
\]

\[
\hat{\ln}(r) = \mu(\ln(r),t)\hat{t} + \sigma(t)\hat{Z}(t) \tag{A.2}
\]

The addition of making volatility a function of time in the short rate model does not change much about the general scope of the model. Changes in the short rate, \(r\), in Equations A.1 and A.2 still evolve randomly through time as the well-known geometric Brownian motion diffusion process. The change does add some complexity to the calibration process since now the model must reflect both the current yield curve and the volatility curve as defined by the modeler. The calibration process will be briefly summarized in its discrete-time form.

Equation A.3 is the discrete-time form of A.2.

\[
\Delta\ln(r) = \mu(\ln(r),t)\Delta t + \sigma(t)\sqrt{\Delta t} \epsilon[\sqrt{\Delta t} \epsilon[-1, +1] \tag{A.3}
\]

\(\epsilon = Z_t - Z_{t+1}\) is the discrete-time version of \(\hat{Z}(t)\) in equation 3.1, also known as a random walk. \(\epsilon\) follows a binomial distribution taking on values of -1 or 1 with probability of 0.50 (as in flipping
an unbiased coin). According to A.3, the short interest rate can only move up or down at any point in the model with equal probability. For example at current time, \( t = 0 \), the current short rate, \( r_0 \), can move up to \( r_u \) or down to \( r_d \), as shown in equations A.4 and A.5.

\[
\ln(r_u) = \ln(r_0) + \mu(\ln(r_0), t)\Delta t + \sigma(t)\sqrt{\Delta t} \tag{A.4}
\]

\[
\ln(r_d) = \ln(r_0) + \mu(\ln(r_0), t)\Delta t - \sigma(t)\sqrt{\Delta t} \tag{A.5}
\]

Combining equations A.4 and A.5 yields a relationship between \( r_u \) and \( r_d \) at any time step in the interest rate tree.

\[
\ln(r_u) - \ln(r_d) = 2 \sigma(t)\sqrt{\Delta t}
\]

\[
\exp(\ln(r_u) - \ln(r_d)) = \exp(2 \sigma(t)\sqrt{\Delta t})
\]

\[
\frac{r_u}{r_d} = \exp(2 \sigma(t)\sqrt{\Delta t})
\]

\[
r_u = r_d \exp(2 \sigma(t)\sqrt{\Delta t}) \tag{A.6}
\]

Equation A.6 determines the basis for the interest rate calibration process, but there also needs to be a calibration for the volatility curve. Rearranging A.6 provides us with a starting point for the volatility calibration at each time step in the interest rate tree.

\[
\ln(r_u) - \ln(r_d) = 2 \sigma(t)\sqrt{\Delta t}
\]

\[
\ln\left(\frac{r_u}{r_d}\right) = 2 \sigma(t)\sqrt{\Delta t}
\]

\[
\sigma(t) = \frac{\ln\left(\frac{r_u}{r_d}\right)}{2 \sqrt{\Delta t}} \tag{A.7}
\]
Equation A.7 must now be incorporated into the calibration process outlined in Section 4. This exercise is left for the reader, but figure A.1 displays short rate tree output for the original BDT paper with the following inputs:

<table>
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<tr>
<th>Maturity (years)</th>
<th>Yield</th>
<th>Yield Volatility</th>
</tr>
</thead>
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<tr>
<td>1.0</td>
<td>10.00%</td>
<td>20.00%</td>
</tr>
<tr>
<td>2.0</td>
<td>11.00%</td>
<td>19.00%</td>
</tr>
<tr>
<td>3.0</td>
<td>12.00%</td>
<td>18.00%</td>
</tr>
<tr>
<td>4.0</td>
<td>12.50%</td>
<td>17.00%</td>
</tr>
<tr>
<td>5.0</td>
<td>13.00%</td>
<td>16.00%</td>
</tr>
</tbody>
</table>

Table A.1. BDT Model Inputs

Figure A.1 BDT Interest Rate Tree